Problem I–1

Let $\mathbb{Q}^+$ denote the set of all positive rational numbers and let $\alpha \in \mathbb{Q}^+$. Determine all functions $f\colon \mathbb{Q}^+ \to (\alpha, +\infty)$ satisfying

$$f\left(\frac{x+y}{\alpha}\right) = \frac{f(x)+f(y)}{\alpha}, \qquad \text{for all } x, y \in \mathbb{Q}^+.$$

Problem I–2

The two figures depicted below consisting of 6 and 10 unit squares, respectively, are called *staircases*.

Consider a 2018×2018 board consisting of 2018^2 cells, each being a unit square. Two arbitrary cells were removed from the same row of the board. Prove that the rest of the board cannot be cut (along the cell borders) into staircases (possibly rotated).

Problem I–3

Let ABC be an acute-angled triangle with $AB < AC$, and let D be the foot of its altitude from A. Let R and Q be the centroids of the triangles ABD and ACD, respectively. Let P be a point on the line segment BC such that $P \neq D$ and the points P, Q, R and D are concyclic. Prove that the lines AP, BQ and CR are concurrent.

Problem I–4

(a) Prove that for every positive integer m there exists an integer $n \geq m$ such that

$$\left\lfloor \frac{n}{1} \right\rfloor \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdots \left\lfloor \frac{n}{m} \right\rfloor = \binom{n}{m}. \qquad (*)$$

(b) Denote by $p(m)$ the smallest integer $n \geq m$ such that the equation $(*)$ holds. Prove that $p(2018) = p(2019)$.

Remark: For a real number x, we denote by $\lfloor x \rfloor$ the largest integer not larger than x.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.